Antiderivative and Area

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1 Antiderivative

Definition 1. A function $F(x)$ is called an antiderivative of $f(x)$ on an interval $I = (a, b)$ if $F'(x) = f(x)$ on this interval.

Example 2. *If $f(x) = x^2$, then $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x)$.*

Also note that for any constant C, $F(x) = \frac{x^3}{3} + C$ is also an antiderivative of $f(x)$.

The above observation is true in general:

Proposition 3. *If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then $F(x) + C$ is also an antiderivative of $f(x)$ on I for any C, and any antidetivative of $f(x)$ on I is of this form.*

The antiderivatives of some basic functions are given below:

Function	$x^n, (n \neq -1)$	$\frac{1}{x}, (x > 0)$	e^x	0
Antiderivative	$\frac{x^{n+1}}{n+1} + C$	$\ln x + C$	$e^x + C$	C

Example 4. *Find the most general antiderivative of $f(x) = \frac{1}{x^2}$, $x > 0$*

If $F(x) = -\frac{1}{x}$, we see $F'(x) = \frac{1}{x^2}$. So the most general antiderivative is of the form $F(x) = -\frac{1}{x} + C$.

Proposition 5. *If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then:*

(i). $cF(x)$ is an antiderivative of $cf(x)$ for any constant c

(ii). $F(x) \pm G(x)$ is an antiderivative of $f(x) \pm g(x)$

Example 6. *Find f if $f'(x) = e^x + 4x$ and $f(0) = 3$.*

f is the antiderivative of f', so f is of the form $f(x) = e^x + 2x^2 + C$ for some constant C.

$3 = f(0) = e^0 + 2 \times 0^2 + C = 1 + C$, we get $C = 2$, so we conclude $f(x) = e^x + 2x^2 + 2$

Example 7. *Find $f(x)$ if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.*

$f'(x) = 4x^3 + 3x^2 - 4x + C_1$, and $f(x) = x^4 + x^3 - 2x^2 + C_1x + C_2$.

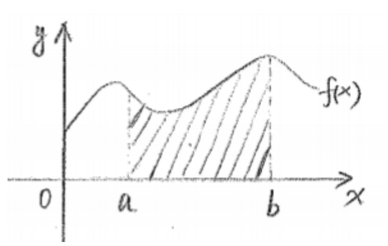
$$\begin{cases} 4 = f(0) = C_2 \\ 1 = f(1) = C_1 + C_2 \end{cases}$$

We get $C_1 = -3$ and $C_2 = 4$, so $f(x) = x^4 + x^3 - 2x^2 - 3x + 4$

In general, when the function is more complicated, it is harder to see what its antiderivative is directly. For example, $f(x) = xe^x$. We will see that the question of finding antiderivative can be transformed to that of integration, thanks to the Fundamental Theorem of Calculus.

2 Area

We are going to define the concept of area for a region bounded between the curve of a function $f(x)$ and x-axis on an interval $[a, b]$.



It seems not easy to come up with a general formula for defining the area, but we can start by thinking about what we know:

If the function is a constant function $f(x) \equiv C$ on $[a, b]$, then we know what the area is: just the area of the corresponding rectangle, $C(b - a)$.

This motivates us the following construction:

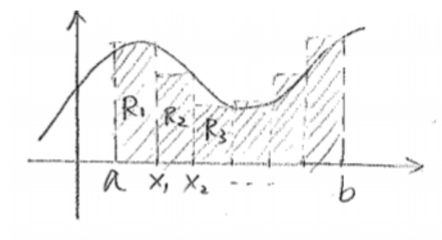
Given a natural number n, we divide $[a,b]$ into n pieces, each of which has length $\Delta x = \frac{b-a}{n}$, so the endpoints of the small pieces are

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, ..., x_n = a_n \Delta x = b$$

Now considering the rectangles with base $[x_{i-1}, x_i]$ and height $f(x_i)$ for all $1 \leq i \leq n$. We sum up the areas of these rectangles:

$$A = f(x_1)\Delta x + ... + f(x_n)\Delta x = (f(x_1) + ... + f(x_n))\Delta x$$

As $n \to \infty$, we see the region covered by these small rectangles converges to the region between the graph of $y = g(x)$ and x-axis.



So we make the following definition:

Definition 8. The area A of the region that lies under the graph of a continuous function f and above x-axis on the interval $[a,b]$ is the limit

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (f(x_1) + ... + f(x_n))\Delta x$$

It can be proved that if we consider the rectangles above $[x_{i-1}, x_i]$ with height $f(x_{i-1})$ instead of $f(x_i)$, then going through the above procedures will lead to the same result, i.e.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} (f(x_0) + ... + f(x_{n-1}))\Delta x$$

Now we introduce a new notation to sum up a sequence: If $a_1, a_2, ..., a_n$ is a sequence, we will denote their sum by

$$\sum_{i=1}^{n} a_i = a_1 + ...a_n$$

So we can write

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Example 9. *Compute the area A of the region that lies below $f(x) = x^2$, above x-axis, and on the interval $[0, 1]$, using both R_n and L_n methods. (We need to make use of the algebraic formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$)*

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$, *so* $x_0 = 0, x_1 = \frac{1}{n}, ..., x_n = \frac{n}{n} = 1.$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} (\frac{i}{n})^2 \Delta x_i = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

So

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{1}{6}(1 + \frac{1}{n})(2 + \frac{1}{n}) = \frac{1}{3}$$

Next we use L_n to compute:

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x_i = \sum_{i=0}^{n-1} (\frac{i}{n})^2 \Delta x_i = \frac{1}{n^3} \sum_{i=0}^{n-1} i^2 = \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2}$$

So

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to\infty} \frac{1}{6}(1 - \frac{1}{n})(2 - \frac{1}{n}) = \frac{1}{3}$$